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The Convergence of a Random Distribution Function Associated with a Branching Process*

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1. Introduction

This paper presents an extension of the results of [1] to age-dependent branching processes. At the same time (and without much additional difficulty) we deal with a branching process on a general state space, rather than with the special model of the binary cascade treated in [1].

We start with a "standard" age-dependent process, defined as in chapter VI of Harris [2]. An initial "parent" particle splits after a random time T into a random number of "offspring" particles. Each of the offspring acts independently as a parent, and after random times (independently distributed as T) produce the next generation of offspring, etc. Let N_t denote the number of particles existing at time t.

The process to be studied here is constructed from the standard one by associating with each particle a "type," namely, a point $\mathbf{x}$ in a d-dimensional Euclidian space Ω. Thus at any given time, each particle existing at that time is to be considered as located at a point in Ω. (In various applications the coordinates of $\mathbf{x}$ will be such quantities as the energy, size, age, location of the associated particle.) Our purpose is to study the diffusion of the particles throughout Ω. Let A be a subset of Ω, $N_t(A)$ be the number of particles in A at time t, and $M_t(A) = N_t(A)/N_t$ be the proportion of particles in A at t. Note that $M_t(\cdot)$ is a random measure; i.e., for each sample path (realization) of the branching process, $M_t(\cdot)$ is a measure for each t. To obtain a nondegenerate limit law we shall let the set A vary with time, and consider a process of the form $M_t(A_t)$. We shall show that by letting A_t grow in a suitable manner, we can attain the convergence (in mean square) of $M_t(A_t)$ to a Gaussian probability function. This, essentially, is the content of Theorem 3 and Remark 3 below.

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2. Definitions and Preliminaries

Let $P\{-\}$ denote the probability of the statement in brackets. Bold face symbols will denote points in Ω; e.g., $\mathbf{x} = (x_1, ..., x_d)$. Define the following random variables and distribution functions:

(i) The time from the birth to the splitting of an arbitrary particle is a random variable T. Let $P\{T \leqslant t\} = G(t)$.

(ii) Given that the particle splits, let J be the number of offspring produced. Let $P\{J = j\} = q(j)$, $\nu = \Sigma jq(j)$, and assume that $\nu > \infty$.

(iii) Let the random variable $\mathbf{X}_0$ denote the associated type vector of a particle. Given that this particle has split into j particles, let the random variables $\mathbf{X}^{(1)}, ..., \mathbf{X}^{(j)}$ denote the associated type vectors of the resultant particles. Let

$$\Phi_j(\mathbf{x}^{(1)}, ..., \mathbf{x}^{(j)} \mid \mathbf{x}_0) = P\{\mathbf{X}^{(1)} \leqslant \mathbf{x}^{(1)}, ..., \mathbf{X}^{(j)} \leqslant \mathbf{x}^{(j)} \mid \mathbf{X}_0 = \mathbf{x}_0, J = j\}$$

denote the conditional joint distribution of the location of the offspring in Ω.

(Note that implicit in the above definitions is the usual assumption for a branching process that each particle behaves independently of the history of the process, and of other particles existing at the same time.)

(iv) Let $N_t(\mathbf{x} \mid \mathbf{x}_0)$ be the number of particles in the set $\Omega(\mathbf{x}) = \{\mathbf{a} : a_i \leqslant x_i, i = 1, ..., d; \mathbf{a} \in \Omega\}$, given that there was a single particle at $\mathbf{x}_0$ at time $t = 0$. Let $p_n(\mathbf{x}, t \mid \mathbf{x}_0) = P\{N_t(\mathbf{x} \mid \mathbf{x}_0) = n\}$.

Using the law of total probability one may argue heuristically that the event "n particles in $\Omega(\mathbf{x})$ at t" can occur in the following mutually exclusive ways: the original particle splits at some time $\tau \leqslant t$ into j particles which move to $\mathbf{x}^{(1)}, ..., \mathbf{x}^{(j)}$; these particles proceed to multiply independently, and in the remaining time $t - \tau$ produce a total of n particles in $\Omega(\mathbf{x})$. Summing over $\tau, j, \mathbf{x}^{(1)}, ..., \mathbf{x}^{(j)}$, and taking into account the additional special circumstances that may prevail when $n = 0$ or 1, we are led to

$$\begin{aligned} p_n(\mathbf{x}, t \mid \mathbf{x}_0) = {} & [1 - G(t)][\delta_{1n} D(\mathbf{x} - \mathbf{x}_0) + \delta_{0n}(1 - D(\mathbf{x} - \mathbf{x}_0))] + q(0)\delta_{0n} G(t) \\ & + \sum_{j=1}^{\infty} q(j) \int_0^t G(d\tau) \int_\Omega \cdots \int_\Omega \Phi_j(d\mathbf{x}^{(1)}, ..., d\mathbf{x}^{(j)} \mid \mathbf{x}_0) \qquad (2.1) \\ & \times p_n(\mathbf{x}, t - \tau \mid \mathbf{x}^{(1)}) * \cdots * p_n(\mathbf{x}, t - \tau \mid \mathbf{x}^{(j)}), \\ & \qquad n = 0, 1, ..., \end{aligned}$$

where

$$D(\mathbf{x}) = \begin{cases} 1 \text{ if } x_i \geqslant 0, & i = 1, ..., d \\ 0 \text{ otherwise,} \end{cases}$$

δ_{ij} is the Kronecker delta, and $*$ here denotes the convolution with respect to the subscript n; e.g.,

$$p_n(\mathbf{x}, t \mid \mathbf{x}^{(1)}) * p_n(\mathbf{x}, t \mid \mathbf{x}^{(2)}) = \sum_{i=0}^{n} p_i(\mathbf{x}, t \mid \mathbf{x}^{(1)}) p_{n-i}(\mathbf{x}, t \mid \mathbf{x}^{(2)}).$$

Associated with (2.1) is an equation for its generating function

$$Q(\theta, \mathbf{x}, t \mid \mathbf{x}_0) = \sum_{n=0}^{\infty} e^{\theta n} p_n(\mathbf{x}, t \mid \mathbf{x}_0), \ (\theta \leqslant 0) \tag{2.2}$$

namely

$$Q(\theta, \mathbf{x}, t \mid \mathbf{x}_0) = [1 - G(t)][e^{\theta} D(\mathbf{x} - \mathbf{x}_0) + (1 - D(\mathbf{x} - \mathbf{x}_0))] + q(0)G(t)$$
$$+ \sum_{j=1}^{\infty} q(j) \int_0^t G(d\tau) \int_{\Omega} \cdots \int_{\Omega} \Phi_j(d\mathbf{x}^{(1)}, \ldots, d\mathbf{x}^{(j)} \mid \mathbf{x}_0) \prod_{i=1}^{j} Q(\theta, \mathbf{x}, t - \tau \mid \mathbf{x}^{(i)}). \tag{2.3}$$

Instead of giving a rigorous derivation of (2.1) and (2.3) we take these equations to be our formal starting point, and state:

THEOREM 1. *The set of equations* (2.1) *has a unique bounded solution* $\{p_n(\mathbf{x}, t \mid \mathbf{x}_0), n = 0, 1, \ldots\}$. *This solution is a probability function, i.e.,* $\sum p_n = 1, p_n \geqslant 0$. $Q(\theta, \mathbf{x}, t \mid \mathbf{x}_0)$ *as defined in* (2.2) *is the unique bounded solution of* (2.3).

A similar theorem was proved for a somewhat more restricted situation in [3] (see Theorem 1 of that paper). The proof of that theorem carries over to the present case with only trivial modifications, and hence we shall not repeat the argument here.

We shall now make our first essential assumption, namely, that for any $j \geqslant 1$ and any $\mathbf{x}_0, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(j)} \in \Omega$,

$$\Phi_j(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(j)} \mid \mathbf{x}_0) = \Phi_j(\mathbf{x}^{(1)} - \mathbf{x}_0, \ldots, \mathbf{x}^{(j)} - \mathbf{x}_0 \mid \mathbf{0}), \tag{2.4}$$

where $\mathbf{0} = (0, \ldots, 0) \in \Omega$. We shall refer to this condition as *spatial homogeneity*, and shall assume that it holds throughout the rest of the paper.

If this condition is satisfied, then direct substitution in (2.1) shows that the latter is also satisfied by $p_n(\mathbf{x} - \mathbf{a}, t \mid \mathbf{x}_0 - \mathbf{a})$ for any $\mathbf{a} \in \Omega$, and hence by the uniqueness part of Theorem 1,

$$p_n(\mathbf{x}, t \mid \mathbf{x}_0) = p_n(\mathbf{x} - \mathbf{x}_0, t \mid \mathbf{0}).$$

Writing $\Phi_j(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(j)} \mid \mathbf{0}) = \Phi_j(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(j)})$, $N(\mathbf{x}, t \mid \mathbf{0}) = N(\mathbf{x}, t)$,

$p_n(\mathbf{x}, t \mid \mathbf{0}) = p_n(\mathbf{x}, t)$ and $Q(\theta, \mathbf{x}, t \mid \mathbf{0}) = Q(\theta, \mathbf{x}, t)$, we see that these functions satisfy:

$$\begin{aligned} p_n(\mathbf{x}, t) = {} & [1 - G(t)][\delta_{1n} D(\mathbf{x}) + \delta_{0n}(1 - D(\mathbf{x}))] + q(0)\delta_{0n} G(t) \\ & + \sum_{j=1}^{\infty} q(j) \int_0^t G(d\tau) \int_\Omega \cdots \int_\Omega \Phi_j(d\mathbf{x}^{(1)}, \ldots, d\mathbf{x}^{(j)}) \\ & \times p_n(\mathbf{x} - \mathbf{x}^{(1)}, t - \tau) * \cdots * p_n(\mathbf{x} - \mathbf{x}^{(j)}, t - \tau), \\ & \qquad n = 0, 1, \ldots; \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} Q(\theta, \mathbf{x}, t) = {} & [1 - G(t)][e^\theta D(\mathbf{x}) + (1 - D(\mathbf{x}))] + q(0) G(t) \\ & + \sum_{j=1}^{\infty} q(j) \int_0^t G(d\tau) \int_\Omega \cdots \int_\Omega \Phi_j(d\mathbf{x}^{(1)}, \ldots, d\mathbf{x}^{(j)}) \\ & \times \prod_{i=1}^{j} Q(\theta, \mathbf{x} - \mathbf{x}^{(i)}, t - \tau). \end{aligned} \tag{2.6}$$

That these equations have unique solutions follows as before and we thus have:

COROLLARY 1. *If* (2.4) *is satified then* (2.5) *has a unique bounded solution* $\{p_n(\mathbf{x}, t), n = 0, 1, \ldots\}$, *which is a probability function.* $Q(\theta, \mathbf{x}, t) = \sum_{n=0}^{\infty} e^{\theta n} p_n(\mathbf{x}, t)$, $(\theta \leqslant 0)$ *is the unique bounded solution of* (2.6).

3. THE MAIN LIMIT THEOREM

We shall assume that G is absolutely continuous with a density $g(\cdot)$, has finite mean and variance; and further that its characteristic function $\varphi(s) = \int e^{ist}\, dG(t)$ satisfies $|\varphi(s)| = 0(|s|^{-\beta})$ for some $\beta > 0$. Following Smith [4] we shall denote by J″ the class of all distributions satisfying these properties. We shall also assume that $\nu_1 = \sum jq(j) > 1$, and $\nu_2 = \sum j(j-1)q(j) < \infty$. Clearly there is then a unique $\alpha > 0$ such that

$$\nu_1 \int_0^\infty e^{-\alpha t} g(t)\, dt = 1. \tag{3.1}$$

Let

$$G^{(\alpha)}(t) = \int_0^t \nu_1 e^{-\alpha x} g(x)\, dx, \quad g^{(\alpha)}(t) = \nu_1 e^{-\alpha t} g(t),$$

$$\mu_\alpha = \int t g^{(\alpha)}(t)\, dt, \quad \sigma_\alpha^2 = \int (t - \mu_\alpha)^2 g^{(\alpha)}(t)\, dt.$$

Note that μ_α and σ_α^2 always exist. Furthermore define

$$\Phi_{ij}(\mathbf{x}) = \Phi_j(\infty, ..., \infty, \mathbf{x}, \infty, ..., \infty),$$

$\mathbf{x}$ being the ith coordinate, and

$$F(\mathbf{x}) = \nu_1^{-1} \sum_{j=1}^{\infty} q(j) \sum_{i=1}^{j} \Phi_{ij}(\mathbf{x}).$$

Note that $F(\cdot)$ is a legitimate distribution function on Ω. Let $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ denote the mean vector and covariance matrix, respectively, of $F(\cdot)$. Recall that $\Phi(\cdot \mid \mathbf{m}, \boldsymbol{\Sigma})$ denotes the multivariate normal distribution with mean vector $\mathbf{m}$ and covariance matrix $\boldsymbol{\Sigma}$. Let A' denote the transpose of a matrix A.

THEOREM 2. *If* $G \in J''$, $\nu_1 > 1, \nu_2 < \infty$, μ *and* $\boldsymbol{\Sigma}$ *exist, and if* $\mathbf{x}_t = (\boldsymbol{\mu}/\mu_\alpha)t + (\boldsymbol{\gamma}/\mu_\alpha^{1/2})t^{1/2}$ *where* $\boldsymbol{\gamma} \in \Omega$, *then as* $t \to \infty$

$$e^{-\alpha t}[N_t(\mathbf{x}_t) - N_t\Phi(\boldsymbol{\gamma} \mid \mathbf{0}, \boldsymbol{\Sigma}^*)] \to 0 \tag{3.2}$$

in mean square, where $\boldsymbol{\Sigma}^* = \boldsymbol{\Sigma} + (\sigma_\alpha/\mu_\alpha)^2\boldsymbol{\mu}'\boldsymbol{\mu}$.

The proof is broken into lemmas dealing with the moments of the processes. Let $\mu(\mathbf{x}, t) = \sum_{n=0}^{\infty} np_n(\mathbf{x}, t)$ and $\mu(t) = \mu(\infty, t)$.

LEMMA 1. *Under the conditions of Theorem* 2

$$\lim_{t\to\infty} e^{-\alpha t}\mu(\mathbf{x}_t, t) = \frac{\nu_1 - 1}{\alpha\nu_1\mu_\alpha} \Phi(\boldsymbol{\gamma} \mid \mathbf{0}, \boldsymbol{\Sigma}^*). \tag{3.3}$$

PROOF. From (2.5) or (2.6) one obtains

$$\mu(\mathbf{x}, t) = [1 - G(t)]D(\mathbf{x}) + \nu_1 \int_0^t g(\tau)\, d\tau \int_\Omega F(d\boldsymbol{\eta})\mu(\mathbf{x} - \boldsymbol{\eta}, t - \tau). \tag{3.4}$$

Note that $\mu(\mathbf{x}, t) \leqslant \mu(t)$ by definition, and hence since $\mu(t)$ always exists under our assumptions (see [2]), so does $\mu(\mathbf{x}, t)$. The passage from (2.6) to (3.4) is formally accomplished by differentiating with respect to θ and letting $\theta \to 0$. The details of the argument are analogous to those found in Chapter VI of [2].

Letting $\mu^{(\alpha)}(\mathbf{x}, t) = e^{-\alpha t}\mu(\mathbf{x}, t)$, we can reduce (3.4) to

$$\mu^{(\alpha)}(\mathbf{x}, t) = e^{-\alpha t}[1 - G(t)]D(\mathbf{x}) + \int_0^t g^{(\alpha)}(\tau)\, d\tau \int_\Omega F(d\boldsymbol{\eta})\mu^{(\alpha)}(\mathbf{x} - \boldsymbol{\eta}, t - \tau). \tag{3.5}$$

From (3.5) one then shows that

$$\mu^{(\alpha)}(\mathbf{x}, t) = \sum_{n=0}^{\infty}\{(e^{-\alpha t}[1 - G(t)]) * g_n^{(\alpha)}(t)\}F_n(\mathbf{x}), \tag{3.6}$$

where $F_0 = D$, $*$ denotes the ordinary convolution operation, and $g_0^{(\alpha)} * f = f$ for any function f. Furthermore one can show that (i) the series in (3.6) converges uniformly in t and in $\mathbf{x}$ for $\mathbf{x}$ in a bounded set $B \subset \Omega$; and that (ii) $\mu^{(\alpha)}(\mathbf{x}, t)$ as expressed in (3.6) is the unique solution of (3.5) which is bounded over all bounded $(\mathbf{x}, t)$-sets. The arguments used in establishing these facts are very similar to those used in the proof of Theorem 1, and the reader is again referred to [3] to see the kind of reasoning involved.

Now let $\quad e^{-\alpha t}[1 - G(t)] = \hat{g}(t)$,
and

$$B^+(t) = \mu_\alpha^{-1}t + \sigma_\alpha\mu_\alpha^{-3/2}Bt^{1/2},$$

$$B^-(t) = \mu_\alpha^{-1}t - \sigma_\alpha\mu_\alpha^{-3/2}Bt^{1/2}.$$

Decompose $\mu^{(\alpha)}(\mathbf{x}, t)$ as follows:

$$\mu^{(\alpha)}(\mathbf{x}, t) = \sum_{n=B^-(t)}^{B^+(t)} + \sum_{\substack{n<B^-(t)\\ n>B^+(t)}} [\hat{g}(t) * g_n^{(\alpha)}(t)]F_n(\mathbf{x}). \tag{3.7}$$

Denote the first sum on the right by $\nu_1^{(\alpha)}(\mathbf{x}, t)$ and the second by $\nu_2^{(\alpha)}(\mathbf{x}, t)$.

We note first that

$$\nu_2^{(\alpha)}(\mathbf{x}, t) \leqslant e^{-\alpha t}[1 - G(t)] * \sum_{\substack{n<B^-(t)\\ n>B^+(t)}} g_n^{(\alpha)}(t). \tag{3.8}$$

We claim that

$$\lim_{t\to\infty} \sum_{\substack{n<B^-(t)\\ n>B^+(t)}} g_n^{(\alpha)}(t) = o_B(1) \tag{3.9}$$

where $o_B(1) \to 0$ as $B \to \infty$. To show this, we use the following local central limit theorem (Smith [4]): Let $f(x)$ be a density function in the class J''. Let $f_n(x)$ denote the n-fold convolution of f, and $\varphi(x \mid \mu, \sigma^2)$ denote the Gaussian density with mean μ and variance σ^2. Then

$$\lim_{n\to\infty} | x |^m\{\sigma \sqrt{n} f_n(n\mu + x\sigma \sqrt{n}) - \varphi(x \mid 0,1)\} = 0 \tag{3.10}$$

for $0 \leqslant m \leqslant 2$, uniformly in $\mathbf{x}$. Our assumptions on G enable us to apply (3.10) with $m = 2$, and one easily shows (see e.g., Cox and Smith [5]) that there is a $C < \infty$ such that for n sufficiently large, say $> n_0$,

$$g_n^{(\alpha)}(t) < \frac{C\sigma_\alpha \sqrt{n}}{(t - n\mu_\alpha)^2} .$$

Then

$$\lim_{t\to\infty} \sum_{\substack{n < B^-(t) \\ n > B^+(t)}} g_n^{(\alpha)}(t) < \lim_{t\to\infty} C\sigma_\alpha \sum_{\substack{n < B^-(t) \\ n > B^+(t)}} \frac{\sqrt{n}}{(t - n\mu_\alpha)^2},$$

and the right side of this inequality can be made arbitrarily small by taking B large. This proves (3.9). Since $e^{-\alpha t}[1 - G(t)]$ is bounded and integrable, it also follows by an elementary argument from (3.8) that

$$\lim_{t\to\infty} \nu_2^{(\alpha)}(\mathbf{x}, t) \leqslant o_B(1) \tag{3.11}$$

uniformly in $\mathbf{x}$.

We turn next to $\nu_1^{(\alpha)}(\mathbf{x}_t, t)$, where $\mathbf{x}_t$ is as defined in the theorem. Applying (3.10) with $m = 0$ we see that for any $\epsilon > 0$ there is a $t_0 < \infty$ and a sequence $\{\delta_n\}$ with $|\delta_n| < \epsilon$, such that for $t > t_0$

$$\begin{aligned} \nu_1^{(\alpha)}(\mathbf{x}_t, t) = \sum_{n=B^-(t)}^{B^+(t)} \left[\int_0^t \hat{g}(z) \frac{1}{\sqrt{n}\,\sigma_\alpha} \varphi\left(\frac{t - z - n\mu_\alpha}{\sigma_\alpha \sqrt{n}} \,\middle|\, 0,1\right) dz\right] \\ \times F_n\left(\frac{\boldsymbol{\mu}}{\mu_\alpha} t + \frac{\boldsymbol{\Upsilon}}{\mu_\alpha^{1/2}} t^{1/2}\right) + \sum_{n=B^-(t)}^{B^+(t)} \frac{\delta_n}{\sqrt{n}} . \end{aligned} \tag{3.13}$$

For fixed t the sum and integral in the first term on the right side of (3.13) are both finite and can be interchanged. We can thus write

$$\nu_1^{(\alpha)}(\mathbf{x}_t, t) = \int_0^t \hat{g}(z)\psi_t(z)\, dz + 2\,\delta\mu_\alpha^{-1}\,\sigma_\alpha B \tag{3.14}$$

where

$$\psi_t(z) = \sum_{n=B^-(t)}^{B^+(t)} \frac{1}{\sqrt{n}\,\sigma_\alpha} \varphi\left(\frac{t - z - n\mu_\alpha}{\sigma_\alpha \sqrt{n}} \,\middle|\, 0,1\right) F_n\left(\frac{\boldsymbol{\mu}}{\mu_\alpha} t + \frac{\boldsymbol{\Upsilon}}{\mu_\alpha^{1/2}} t^{1/2}\right),$$

and $|\delta| < \epsilon$. We claim that

$$\lim_{t\to\infty} \psi_t(z) = \mu_\alpha^{-1}\Phi\left(\boldsymbol{\Upsilon} \,|\, 0; \boldsymbol{\Sigma} + \left(\frac{\sigma_\alpha}{\mu_\alpha}\right)^2 \boldsymbol{\mu}'\boldsymbol{\mu}\right) + o_B(1) \tag{3.15}$$

uniformly for $0 \leqslant z \leqslant t_1$, for any $t_1 < \infty$. To show this, let $v_n = (n\mu_\alpha - t)/\sigma_\alpha \sqrt{n}$, and note that by the standard central limit theorem

$$\lim_{n\to\infty} F_n(\mathbf{x}) = \lim_{n\to\infty} \Phi\left(\frac{\mathbf{x} - n\boldsymbol{\mu}}{\sqrt{n}} \,\Big|\, 0, \Sigma\right)$$

uniformly in $\mathbf{x}$. Then

$$\lim_{t\to\infty} \psi_t(z) = \lim_{t\to\infty} \sum_{n=B^-(t)}^{B^+(t)} \frac{1}{\sqrt{n}\,\sigma_\alpha} \varphi\left(v_n - \frac{z}{\sigma_\alpha \sqrt{n}} \,\Big|\, 0,1\right) \Phi\left(\boldsymbol{\gamma} - \frac{\boldsymbol{\mu}\sigma_\alpha}{\mu_\alpha} v_n \,|\, 0, \Sigma\right).$$

Noting that $v_{n+1} - v_n \sim n^{-1/2}\mu_\alpha\sigma_\alpha^{-1}$ we see that

$$\lim_{t\to\infty} \psi_t(z) = \int_{-B}^{B} \varphi(v)\, \Phi\left(\boldsymbol{\gamma} - \frac{\boldsymbol{\mu}\sigma_\alpha}{\mu_\alpha} v \,\Big|\, 0, \Sigma\right) \frac{dv}{\mu_\alpha},$$

which implies (3.15).

Now $\int_0^\infty \hat{g}(z)\, dz = (\nu_1 - 1)/\alpha\nu_1$. Using this fact with (3.15), applying an elementary truncation argument to (3.14), and taking B large and then δ small, shows that

$$\lim_{t\to\infty} \nu_1^{(\alpha)}(\mathbf{x}, t) = (\nu_1 - 1)(\alpha\nu_1\mu_\alpha)^{-1}\Phi(\boldsymbol{\gamma} \,|\, 0; \Sigma^*).$$

Together with (3.11), this implies the lemma.

Let $\mu_2(\mathbf{x}, t) = \sum_{n=0}^\infty n^2 p_n(\mathbf{x}, t)$. Existence of this moment is again implied by that of the corresponding moment of N_t (see [2]). Define

$$H(\mathbf{u}_1, \mathbf{u}_2) = \nu_2^{-1} \sum_{j=1}^\infty q(j) \sum_{\substack{i,k=0\\ i\neq k}}^{j} \Phi_{i,k,j}(\mathbf{u}_1, \mathbf{u}_2),$$

where

$$\Phi_{i,k,j}(\mathbf{u}_1, \mathbf{u}_2) = \Phi_j(\infty, \ldots, \infty, \mathbf{u}_1, \infty, \ldots, \infty, \mathbf{u}_2, \infty, \ldots, \infty),$$

$\mathbf{u}_1$ and $\mathbf{u}_2$ being the ith and kth arguments, respectively, of Φ_j. We next prove

LEMMA 2. *Under the assumptions of Theorem* 2

$$\lim_{t\to\infty} e^{-2\alpha t}\mu_2(\mathbf{x}_t, t) = \left\{\frac{\nu_2 \int_0^\infty e^{-2\alpha\tau} g(\tau)\, d\tau}{1 - \nu_1 \int_0^\infty e^{-2\alpha\tau} g(\tau)\, d\tau}\right\} \left(\frac{\nu_1 - 1}{\alpha\nu_1\mu_\alpha}\right)^2 \Phi^2(\boldsymbol{\gamma} \,|\, 0, \Sigma^*).$$

PROOF. Direct computation from (2.5) or twofold differentiation of (2.6) with respect to θ yields

$$\mu_2(\mathbf{x}, t) = [1 - G(t)]D(\mathbf{x}) + \nu_1 \int_0^t g(\tau)\, d\tau \int_\Omega F(d\eta)\mu_2(\mathbf{x} - \eta, t - \tau) \\ + \nu_2 \int_0^t g(\tau)\, d\tau \int_\Omega \int_\Omega H(d\eta_1, d\eta_2)\mu(\mathbf{x} - \eta_1, t - \tau)\mu(\mathbf{x} - \eta_2, t - \tau). \tag{3.16}$$

(The details of the passage from (2.6) to (3.16) are accomplished as in the case of (3.4).) Setting

$$\mu_2^{(\alpha)}(\mathbf{x}, t) = e^{-2\alpha t}\mu_2(\mathbf{x}, t),$$

and

$$J^{(\alpha)}(\mathbf{x}, t) = e^{-2\alpha t}[1 - G(t)]D(\mathbf{x}) \\ + \nu_2 \int_0^t e^{-2\alpha\tau} g(\tau)\, d\tau \int_\Omega \int_\Omega H(d\eta_1, d\eta_2)\mu^{(\alpha)}(\mathbf{x} - \eta_1, t - \tau) \\ \cdot \mu^{(\alpha)}(\mathbf{x} - \eta_2, t - \tau),$$

we can write

$$\mu_2^{(\alpha)}(\mathbf{x}, t) = J^{(\alpha)}(\mathbf{x}, t) + \int_0^t g^{(2\alpha)}(\tau)\, d\tau \int_\Omega F(d\eta)\mu_2^{(\alpha)}(\mathbf{x} - \eta, t - \tau). \tag{3.17}$$

Now by Lemma 1

$$\lim_{t\to\infty} J^{(\alpha)}(\mathbf{x}_t, t) = \left(\frac{\nu_1 - 1}{\alpha\nu_1\mu_\alpha}\right)^2 \Phi^2(\gamma \mid 0, \Sigma^*)\nu_2 \int_0^\infty e^{-2\alpha\tau} g(\tau)\, d\tau.$$

Denote the right side by J_1. By successive substitution (3.17) yields

$$\mu_2^{(\alpha)}(\mathbf{x}, t) = \sum_{n=0}^{\infty} J^{(\alpha)}(\mathbf{x}, t) \circledast g_n^{(2\alpha)}(t)F_n(\mathbf{x}), \tag{3.19}$$

where the following notation for convolutions is used:

$$g_n^{(\alpha)}(t) = \int_0^t g_{n-1}^{(\alpha)}(t - \tau)g^{(\alpha)}(\tau)\, d\tau,$$

$$F_n(\mathbf{x}) = \int_\Omega F_{n-1}(\mathbf{x} - \eta)\, dF(\eta),$$

and

$$J^{(\alpha)}(\mathbf{x}, t) \circledast g_n^{(\alpha)}(t)F_n(\mathbf{x}) = \int_0^t \int_\Omega J^{(\alpha)}(\mathbf{x} - \eta, t - \tau)g_n^{(\alpha)}(\tau)\, d\tau\, dF_n(\eta).$$

The question of convergence of the series in (3.19) is in this case trivial, since $\int_0^\infty g^{(2\alpha)}(t)\, dt < 1$, and since there is hence a constant $c < 1$ such that

$J^{\alpha}(\mathbf{x}, t) \circledast g_n^{(2\alpha)}(t)F_n(\mathbf{x}) <$ constant $\cdot\, c^n$. That (3.19) satisfies (3.17) is directly verifiable, and the uniqueness of the solution is proved as in the case of the first moment.

We now replace $\mathbf{x}$ by $\mathbf{x}_t$ in (3.19) take the limit as $t \to \infty$, and make use of the uniform convergence of the series to take the limit through the sum. But

$$\lim_{t\to\infty} J^{(\alpha)}(\mathbf{x}_t, t) \circledast g_n^{(2\alpha)}(t)F_n(\mathbf{x}_t) = J_1\left[\nu_1 \int_0^\infty e^{-2\alpha\tau}g(\tau)\, d\tau\right]^n$$

and thus (3.19) implies the lemma.

The last lemma needed is a close analogue of Lemma 2, but for product moments. Let $p_{n,m}(\mathbf{x}, t) = P\{N_t(\mathbf{x}) = n, N_t = m\}$ and

$$m(\mathbf{x}, t) = \sum_{n,m} nmp_{n,m}(\mathbf{x}, t).$$

Then arguing similarly to Lemma 2, or to Lemma 3 of [1], one readily obtains

LEMMA 3. *Under the assumptions of Theorem* 2

$$\lim_{t\to\infty} e^{-2\alpha t}m(\mathbf{x}_t, t) = \left\{\frac{\nu_2 \int_0^\infty e^{-2\alpha\tau}g(\tau)\, d\tau}{1 - \nu_1 \int_0^\infty e^{-2\alpha\tau}g(\tau)\, d\tau}\right\} \left(\frac{\nu_1 - 1}{\alpha\nu_1\mu_\alpha}\right)^2 \Phi(\gamma \mid 0, \Sigma^*). \tag{3.20}$$

We are now ready to proceed with the

PROOF OF THEOREM 2. Let

$$\left\{\frac{\nu_2 \int_0^\infty e^{-2\alpha\tau}g(\tau)\, d\tau}{1 - \nu_1 \int_0^\infty e^{-2\alpha\tau}g(\tau)\, d\tau}\right\} \left[\frac{\nu_1 - 1}{\alpha\nu_1\mu_\alpha}\right]^2 = K^2$$

where clearly $0 < K^2 < \infty$.

To prove the theorem we simply note that

$$\lim_{t\to\infty} E\left[\frac{N_t(\mathbf{x}_t)}{Ke^{\alpha t}\Phi(\gamma \mid 0, \Sigma^*)} - \frac{N_t}{Ke^{\alpha t}}\right]^2 = 0, \tag{3.21}$$

where E denotes expectation. This follows at once by multiplying out the square in (3.21), and applying Lemmas 2 and 3, and the fact that

$$\lim_{t\to\infty} e^{-2\alpha t} \sum_{n=0}^{\infty} n^2 P\{N_t = n\} = K^2; \tag{3.22}$$

which in turn is derived from the relation

$$\mu_2^{(\alpha)}(t) = e^{-2\alpha t}[1 - G(t)] + \nu_2 \int_0^t e^{-2\alpha\tau} g(\tau)\mu^{(\alpha)2}(t-\tau)\, d\tau$$
$$+ \int_0^t g^{(2\alpha)}(\tau)\mu_2^{(\alpha)}(t-\tau)\, d\tau,$$

where

$$\mu_2^{(\alpha)}(t) = e^{-2\alpha t} \sum_{n=0}^{\infty} n^2 P\{N_t = n\}.$$

COROLLARY. *Under the conditions of Theorem 2, $N_t(\mathbf{x}_t)/[\mu(t)\Phi(\boldsymbol{\gamma} \mid \mathbf{0}, \boldsymbol{\Sigma}^*)]$ converges in mean square to a random variable W, which is same as the limit (in mean square) of $N_t/\mu(t)$.*

REMARK 1. It is known (Bellman and Harris [6]) that $N_t/\mu(t)$ converges in mean square to a random variable W, whose distribution is continuous except for a jump at 0. The mangitude of this jump is the smallest nonnegative root of $t = \sum_{j=0}^{\infty} q(j)t^j$. If $1 - G(\tau) = 0(e^{-c\tau})$, $c > 0$, then the distribution is absolutely continuous except for the same jump at 0.

PROOF OF COROLLARY. Just note that

$$E\left[\frac{N_t(\mathbf{x}_t)}{\mu(t)\Phi(\boldsymbol{\gamma} \mid \mathbf{0}, \boldsymbol{\Sigma}^*)} - W\right]^2 \leqslant 2E\left[\frac{N_t(\mathbf{x}_t)}{\mu(t)\Phi(\boldsymbol{\gamma} \mid \mathbf{0}, \boldsymbol{\Sigma}^*)} - \frac{N_t}{\mu(t)}\right]^2$$
$$+ 2E\left[\frac{N_t}{\mu(t)} - W\right]^2.$$

The first term on the right goes to zero due to Theorem 2, and since $e^{-\alpha t}\mu(t) \to$ constant as $t \to \infty$; and the second vanishes due to remark 1.

We are now ready to state our main result. To make it appear most natural we introduce the notion $\Gamma_t(\boldsymbol{\gamma}) = M_t[\Omega(\mathbf{x}_t)] = N_t(\mathbf{x}_t)/N_t$, where $\mathbf{x}_t$ is as defined in Theorem 2. Note that $\Gamma_t(\cdot)$ is a random distribution function in the sense that for each sample path of the process it is (for each t) a distribution function on Ω.

THEOREM 3. *If in addition to the assumptions of Theorem 2 we assume that $q(0) = 0$, then $\Gamma_t(\boldsymbol{\gamma}) \to \Phi(\boldsymbol{\gamma} \mid \mathbf{0}, \boldsymbol{\Sigma}^*)$ in mean square (as $t \to \infty$).*

REMARK 2. The condition $q(0) = 0$ is clearly essential, since otherwise $P\{N_t = 0\} > 0$, and with positive probability $\Gamma_t(\gamma)$ is not defined.

PROOF OF THEOREM 3. Let

$$\frac{N_t(\mathbf{x}_t)}{\mu(t)\Phi(\boldsymbol{\gamma} \mid \mathbf{0}, \boldsymbol{\Sigma}^*)} = A_t\,; \frac{N_t}{\mu(t)} = B_t\,.$$

Then we want to prove that

$$\lim_{t\to\infty} E\left[\frac{A_t}{B_t} - 1\right]^2 = 0.$$

But by definition of A_t, B_t we see at once that $| A_t/B_t - 1 | \leqslant D =$ finite constant, and hence for any $\delta > 0$

$$E\left[\frac{A_t}{B_t} - 1\right]^2 = E\left[\left(\frac{A_t}{B_t} - 1\right)^2 \Big| \left(\frac{A_t}{B_t} - 1\right)^2 \leqslant \delta\right] P\left[\left(\frac{A_t}{B_t} - 1\right)^2 \leqslant \delta\right]$$
$$+ E\left[\left(\frac{A_t}{B_t} - 1\right)^2 \Big| \left(\frac{A_t}{B_t} - 1\right)^2 > \delta\right] P\left[\left(\frac{A_t}{B_t} - 1\right)^2 > \delta\right]$$
$$\leqslant \delta + D^2 P\left[\left|\frac{A_t}{B_t} - 1\right| > \sqrt{\delta}\right].$$

Now it is elementary to show that the mean square convergence of A_t and B_t to a random variable W with a continuous distribution function, implies that

$$P\left[\left|\frac{A_t}{B_t} - 1\right| > \sqrt{\delta}\right] \to 0 \text{ as } t \to \infty$$

for any δ. But the result quoted in Remark 1 assures us that the distribution of W is continuous except at 0, and the assumption $q(0) = 0$ removes the jump at zero, and hence we are finished.

REMARK 3. Using standard measure theoretic arguments one can extend Theorem 3 to state that for any Borel subset A of Ω such that $\Phi(A^i \mid 0, \Sigma^*) = \Phi(\bar{A} \mid 0, \Sigma^*)$ (where A^i and $\bar{A}$ are the interior and closure of A, respectively), we have $M_t(A_t) \to \Phi(A \mid 0, \Sigma^*)$ in mean square, where $A_t = \{\mathbf{x}_t : \gamma \in A\}$.

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